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Stabilization and Random Linear Regulator Problem for Random Linear Control Processes

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We study in a dynamical system context the random feedback stabilization problem for linear, random control processes. We are led to study the random linear regulator problem, which we solve by considering the spectral theory of linear time-dependent Hamiltonian systems. This is done with the aid of the concepts of exponential dichotomy and rotation number. © 1996 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to study in a dynamical systems context the feedback stabilization problem for random linear control processes on the semi-infinite interval. By "random," we mean processes

$$x' = a(t)x + b(t)u \qquad (x \in \mathbf{R}^n, u \in \mathbf{R}^m) \tag{1.1}$$

for which $a(\cdot)$ and $b(\cdot)$ may exhibit the entire range of behaviour from periodicity to (bounded) i.i.d.'s (i.e., independent identically distributed processes). Though this usage is not entirely standard, it is motivated by the meaning of the word "random" in the theory of the random Schrö-

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dinger operator. Stochastic methods, powerful as they are, can only be applied when the coefficients exhibit strong independence properties. Our methods apply in a vast range of circumstances when independence is not present.

In a standard way, the discussion of this problem will lead us to consider the (random) linear regulator problem. This problem is usually solved by constructing a solution of a Riccati equation. Our point of view is that the solution of the Riccati equation is best obtained by showing that the associated linear Hamiltonian system has an ED (exponential dichotomy). This implies very strong robustness properties and very good smoothness properties with respect to parameters, of the solution as well as the feedback stabilizer. We also obtain immediately the important property of "preservation of recurrence" of which more below. Here we note that if $a(t)$ and $b(t)$ exhibit chaotic time dependence, then the feedback control $k(t)$ is "no more chaotic" than a and b [19].

We prove the existence of an exponential dichotomy by using the rotation number of the linear Hamiltonian system. In fact, we show that under a mild controllability hypothesis on (1.1), the rotation number of the linear Hamiltonian system is constant on an interval. This implies the existence of ED, which then by standard results gives the robustness, smoothness, and preservation of recurrence properties mentioned above. We shall also obtain a proper random analog of the pole relocation theorem.

We finish the introduction by describing more precisely our formulation of the random feedback stabilization problem. Let $M(n,m)$ be the set of $n \times m$ real matrices. Fix a number $1 \le p \le \infty$, and let

$$C = \left\{ a \colon \mathbf{R} \to M(n,n) \mid \operatorname*{Sup}_{t \in \mathbf{R}} \int_t^{t+1} \| a(s) \|^p \, ds < \infty \right\}.$$

If $p = 1$, suppose in addition that

$$\lim_{\varepsilon \to 0} \int_t^{t+\varepsilon} \| a(s) \| \, ds = 0$$

uniformly in $t \in \mathbf{R}$. Give C the distribution topology; that is, $a_n \to a$ in C if and only if

$$\int_{\mathbf{R}} a_n(t) \varphi(t) \, dt \to \int_{\mathbf{R}} a(t) \varphi(t) \, dt$$

for each smooth function $\varphi \colon \mathbf{R} \to \mathbf{R}^n$ of compact support. Next define $D = \{ b \colon \mathbf{R} \to M(n,m) \mid b$ is uniformly bounded and uniformly continuous$\}$. Give D the topology of uniform convergence on compact sets. Both C and

D support the *translation flow* $\{\tau_t \mid t \in \mathbf{R}\}$, where τ_t is the translation

$$(\tau_t a)(s) = a(t+s), \quad (\tau_t b)(s) = b(t+s) \qquad (t, s \in \mathbf{R}).$$

We will choose the coefficient functions $a(t), b(t)$ in (1.1) from the sets C, D, respectively. In order to use techniques of dynamical systems, however, we will replace (1.1) by a *family* of control processes having the same form. Thus let Ω be a compact, translation-invariant subset of $C \times D$ (thus in particular Ω is metrizable). The pair $(\Omega, \{\tau_t \mid t \in \mathbf{R}\})$ defines a *flow* on Ω in the sense that

(i) $\tau_0 \colon \Omega \to \Omega$ is the identity map;
(ii) $\tau_t \circ \tau_s = \tau_{t+s}$ $(t, s \in \mathbf{R})$;
(iii) the map $(\omega, t) \to \tau_t(\omega)$ from $\Omega \times \mathbf{R}$ to Ω is jointly continuous.

Each element of Ω gives rise to a control process having the form of (1.1):

$$x' = A(\tau_t(\omega))x + B(\tau_t(\omega))u \qquad (\omega \in \Omega) \qquad (1.1)_\omega$$

We must explain the notation. If $\omega = (a, b) \in \Omega$, define $B(\omega) = b(0)$; then it is easily seen that $B \colon \Omega \to M(n, m)$ is continuous and $B(\tau_t(\omega)) = b(t)$, $(t \in \mathbf{R})$. On the other hand, there may not be a *continuous* function $A \colon \Omega \to M(n, n)$ such that $A(\tau_t(\omega)) = a(t)$ for all $t \in \mathbf{R}$. Therefore we regard the expression "$t \to A(\tau_t(\omega))$" as a notational device for expressing the function $a(t)$. Thus $t \to A(\tau_t(\omega))$ represents the projection of ω on to its first coordinate.

This framework includes *all* time-varying systems with bounded coefficients, from periodic and almost periodic to highly stochastic systems with positive entropy, and also the intermediate case of zero entropy with or without mixing conditions. We use the word "random" to refer to this very wide class of coefficients.

The original control process (1.1) generates a family $(1.1)_\omega$ if $a \in C$ and $b \in D$. Namely, let $\Omega = \operatorname{cls}\{(\tau_t(a), \tau_t(b)) \mid t \in \mathbf{R}\} \subseteq C \times D$. If $\omega_0 = (a, b)$, then Eq. $(1.1)_{\omega_0}$ coincides with (1.1). Our point of view is quite general in that we do not require the existence of a point $\omega_0 \in \Omega$ whose orbit $\{\tau_t(\omega_0) \mid t \in \mathbf{R}\}$ is dense in Ω.

The random stabilization problem is that of finding *a* (at least) continuous map $K \colon \Omega \to M(m, n)$ such that for each $\omega \in \Omega$ the origin is an asymptotically stable fixed point of $(1.1)_\omega$ with feedback rule

$$u_\omega(t) = K(\tau_t(\omega))x(t).$$

We remark that even though results of [7, 14, 20] ensure existence of feedback matrix $K_\omega(t)$ for each ω, it is not clear that the family $\{K_\omega\}$ "lifts consistently” to Ω; i.e., it is not clear that there exists K: $\Omega \to M(m, n)$ such that $K_\omega(t) = K(\tau_t(\omega))$. This problem is non-trivial in general; this is what we mean by “preservation of recurrence properties.” Our techniques immediately yield a consistent lifting.

The strength of our methods is also illustrated when one considers robustness and smoothness of the feedback matrix K. For example if Ω is a manifold and A, B are C^r smooth functions then K is C^r as well. Robustness refers to variation of K when Ω itself is varied (e.g. in some function space); results of Sacker–Sell [27] can be used to show that K is “continuous” under such variation in very general circumstances.

Finally, we also mention that the exponential dichotomy property is very much stronger than that of positive Lyapunov exponents and in this sense our results are stronger than those of Bougerol [5, 6] when the coefficients satisfy the boundedness criterion mentioned above.

The paper is organized as follows. In Section 2, we discuss quite weak conditions which imply uniform controllability. In Section 3, we briefly discuss the random linear regulator problem, introduce the concept of ED, and discuss the rotation number of linear Hamiltonian systems. In the final section, we solve the random feedback stabilization problem.

2. A CONDITION IMPLYING UNIFORM CONTROLLABILITY

We introduce some notation. Consider the family of control processes

$$x' = A(\tau_t(\omega))x + B(\tau_t(\omega))u \qquad (\omega \in \Omega), \tag*{$(1.1)_\omega$}$$

where the flow $(\Omega, \{\tau_t\})$ is as described in Section 1. Let $X(\omega, t)$ be the fundamental matrix solution of the linear system

$$x' = A(\tau_t(\omega))x \qquad (\omega \in \Omega) \tag*{$(2.1)_\omega$}$$

satisfying $X(\omega, 0) = I$—the $n \times n$ identity matrix. Then X: $\Omega \times \mathbf{R} \to \mathrm{GL}(n, \mathbf{R})$ is jointly continuous and satisfies the following “*cocycle identity*”:

$$X(\omega, t+s) = X(\tau_t(\omega), s)X(\omega, t) \qquad (\omega \in \Omega, t, s \in \mathbf{R}).$$

We will also consider the adjoint system

$$x' = -A^*(\tau_t(\omega))x \qquad (\omega \in \Omega) \tag*{$(2.2)_\omega$}$$

(here * denotes the transpose); the fundamental matrix solution of this system is $Z(\omega, t) = X^*(\omega, t)^{-1}$.

Let $\omega \in \Omega$ and $t_1 < t_2 \in \mathbf{R}$. Define the following $n \times n$ matrix-valued function of ω:

$$W_{[t_1, t_2]}(\omega) = \int_{t_1}^{t_2} X(\omega, t)^{-1} B(\tau_t(\omega)) B^*(\tau_t(\omega)) X^*(\omega, t)^{-1} \, dt.$$

We will write $W_T(\omega)$ for $W_{[0,T]}(\omega)$.

LEMMA 2.1. *With the above notation, the following identity holds for all* $t, T \in \mathbf{R}$, *and* $\omega \in \Omega$:

$$X(\omega, t) W_{[t, t+T]}(\omega) X^*(\omega, t) = W_T(\tau_t(\omega)).$$

Proof. This is a simple calculation using the cocycle identity. ∎

DEFINITION 2.2. Fix $\omega \in \Omega$. The process $(1.1)_\omega$ is said to be *controllable* (more precisely null controllable at ω) if and only if given $x_0 \in \mathbf{R}^n$, there exists a locally integrable function $u(t)$ and a number $T > 0$ such that the solution $x(t)$ of $(1.1)_\omega$ with $u = u(t)$ and $x(0) = x_0$ satisfies $x(T) = 0$.

For each $\omega \in \Omega$, it is well known that controllability of the process $(1.1)_\omega$ is equivalent to the non-singularity of $W_T(\omega)$ for some $T > 0$.

DEFINITION 2.3. Fix $\omega \in \Omega$. The process $(1.1)_\omega$ is said to be *uniformly controllable* if and only if there exist constants $\alpha > 0$, $T > 0$ such that

$$0 < \alpha I < W_T(\tau_t(\omega)), \qquad \forall t \in \mathbf{R}.$$

We prove that if the flow $(\Omega, \{\tau_t\})$ is *minimal* in the sense that if the orbit $\{\tau_t(\omega) \mid t \in \mathbf{R}\}$ is dense in Ω for all $\omega \in \Omega$, then uniform controllability on Ω is equivalent to simple controllability along a single orbit. This is a generalization of a theorem of Artstein [3] and is the first step towards our sufficiency condition for uniform controllability (Proposition 2.5). The sufficient condition does not assume minimality but rather requires only controllability along at least one orbit in each minimal subset of Ω.

LEMMA 2.4. *Suppose* $(\Omega, \{\tau_t\})$ *is minimal and suppose that for some* $\omega_0 \in \Omega$, *the process* $(1.1)_{\omega_0}$ *is controllable. Then every process* $(1.1)_\omega$ *is uniformly controllable. In fact, there exists constants* $\alpha > 0$, $T > 0$, *which are independent of* ω, *such that*

$$0 < \alpha I < W_T(\omega) \qquad (\omega \in \Omega).$$

Proof. The proof is similar to that of Theorem (2.10) in [16]. Since the process $(1.1)_{\omega_0}$ is controllable, there exists $S > 0$ such that $W_S(\omega_0)$ is non-singular. Choose $\delta > 0$ such that $\delta < \inf\{\langle W_S(\omega_0)x, x\rangle \mid \|x\| = 1\}$. Since the map $(\omega, x) \to \langle W_S(\omega)x, x\rangle$ is continuous and $\{x \mid \|x\| = 1\}$ is compact, there exists a compact neighbourhood N of ω_0 in Ω such that

$$\frac{\delta}{2} < \inf\left\{\langle W_S(\omega)x, x\rangle \mid \|x\| = 1,\ \omega \in N\right\}. \tag{2.3}$$

Let us write temporarily $\tau(\omega, t)$ for $\tau_t(\omega)$, $(\omega \in \Omega, t \in \mathbf{R})$. By minimality of Ω, we can find $L > 0$ and a sequence $T_n \to -\infty$ such that for each $n \geq 1$:

(i) $$T_{n+1} < T_n; \tag{2.4}$$

(ii) $$|T_{n+1} - T_n| \leq L; \tag{2.5}$$

(iii) $$\omega_n = \tau(\omega_0, T_n) \in N. \tag{2.6}$$

By (2.3), we have for $n \geq 1$:

$$0 < \frac{\delta}{2} < \inf\left\{\langle W_T(\tau(\omega_0, T_n))x, x\rangle \mid \|x\| = 1\right\}. \tag{2.7}$$

Let $t < 0$. We can write $t = T_n + \gamma$ for some $n \geq 1$, where $\gamma \in [-L, 0]$. Then for any x with $\|x\| = 1$, we have

$$\begin{aligned}\langle W_{S+L}(\tau_t(\omega_0))x, x\rangle &= \langle W_{S+L}(\tau(\omega_0, T_n + \gamma))x, x\rangle \\ &= \langle X(\omega_n, \gamma)W_{[\gamma, \gamma+S+L]}(\omega_n)X^*(\omega_n, \gamma)x, x\rangle \\ &= \langle W_{[\gamma, \gamma+S+L]}(\omega_n)y_n, y_n\rangle,\end{aligned}$$

where we have written $y_n = X^*(\omega_n, \gamma)x$. Continuing, we have

$$\langle W_{[\gamma, \gamma+S+L]}(\omega_n)y_n, y_n\rangle \geq \langle W_{\gamma+S+L}(\omega_n)y_n, y_n\rangle,$$

since $[0, \gamma + S + L] \subset [\gamma, \gamma + S + L]$. Thus,

$$\begin{aligned}\langle W_{[\gamma, \gamma+S+L]}(\omega_n)y_n, y_n\rangle &\geq \langle W_S(\omega_n)y_n, y_n\rangle \qquad (\text{because } \gamma + L > 0) \\ &\geq \frac{\delta}{2}\|y_n\|^2.\end{aligned}$$

Letting $\eta = \min\{\|X^*(\omega, \gamma)x\|^2 \mid -L \leq \gamma \leq 0,\ \|x\| = 1,\ \omega \in N\}$, we see that

$$\inf\left\{\langle W_{S+L}(\tau_t(\omega_0))x, x\rangle \mid \|x\| = 1, t \leq 0\right\} \geq \frac{\eta\delta}{2} \geq 0.$$

Since the semi-orbit $\{(\tau_t(\omega_0) \mid t \leq 0\}$ is dense in Ω, we have $W_{T+L}(\omega) \geq \eta\delta/2$ for all $\omega \in \Omega$. This shows that the statement of Lemma 2.4 holds with $T = S + L$. ▮

The lemma applies in particular when (1.1) has almost periodic coefficients $a(t), b(t)$. This is due to the fact that in this case the flow $(\Omega, \{\tau_t\})$ is minimal. If $(\Omega, \{\tau_t\})$ is not minimal, then Lemma 2.4 is false, as simple examples show [3]. We now prove the main result.

PROPOSITION 2.5. *Suppose that, for each minimal subset $M \subseteq \Omega$, there exists at least one $\omega_0 \in M$ such that the process $(1.1)_{\omega_0}$ is controllable. Then each process $(1.1)_\omega$ is uniformly controllable and the constants $T > 0$, $\alpha > 0$ can be chosen independent of $\omega \in \Omega$. In fact, $W_T(\omega) \geq \alpha I$ for each $\omega \in \Omega$.*

Proof. Suppose for contradiction that there exist sequences $\omega_j \in \Omega$, $T_j \to \infty$, $\alpha_j \to 0$ and $x_j \in \mathbf{R}^n$ such that $\|x_j\| = 1$ and $\|W_{T_j}(\omega_j)x_j\| \leq \alpha_j$. We can assume that $\omega_j \to \overline{\omega}$, $x_j \to \bar{x}$ where $\|x\| = 1$. Then clearly $W_T(\overline{\omega})\bar{x} = 0$ for all $T > 0$.

Now however it is easily seen that, given $\varepsilon > 0$, there is a minimal subset $M \subseteq \Omega$, a point $\omega_0 \in M$ and a time $t > 0$ such that $d(\tau_t(\overline{\omega}), \omega_0) < \varepsilon$. Here d is some metric on Ω. By continuity in ω of $W_T(\omega)$ and using Lemma 2.1, we see that there exists $T_* > 0$ such that $\|W_{T_*}(\overline{\omega})\bar{x}\| > 0$. This is a contradiction; the proposition is proved. ▮

3. THE RANDOM LINEAR REGULATOR PROBLEM

We give a brief introduction to the random linear regulator problem and the standard approach to solving it. In doing so, we shall have occasion to recall the "Floquet Theory" and, in particular, the notion of rotation number for linear Hamiltonian systems.

Consider once again the family of control processes

$$x' = A(\tau_t(\omega))x + B(\tau_t(\omega))u \qquad (\omega \in \Omega), \tag*{$(1.1)_\omega$}$$

where $(\Omega, \{\tau_t\})$ is a flow as discussed in Section 1. Introduce continuous matrix-valued functions $Q: \Omega \to M(n,n)$ and $R: \Omega \to M(m,m)$ with the following properties:

$$Q^*(\omega) = Q(\omega) \geq 0 \qquad (\omega \in \Omega) \tag*{$(3.1)_{\mathrm{a}}$}$$

$$R^*(\omega) = R(\omega) > 0 \qquad (\omega \in \Omega). \tag*{$(3.1)_{\mathrm{b}}$}$$

In particular, R is assumed to be strictly positive-definite. Define the

Lagrangian

$$2\tilde{L}_\omega(x,u) = \langle x, Q(\omega)x\rangle + \langle u, R(\omega)u\rangle,$$

and set

$$L_\omega(x,u) = \int_0^\infty 2\tilde{L}_\omega(x(t),u(t))\,dt.$$

The linear regulator problem is the following. Fix an initial vector $x_0 \in \mathbf{R}^n$. We are to find a control function $u(t) = u_\omega(t) \in L^2([0,\infty), \mathbf{R}^m)$ such that, if $x_\omega(t)$ is the solution of $(1.1)_\omega$ with $x_\omega(0) = x_0$ and $u = u_\omega(t)$, then the pair (x_ω, u_ω) minimizes the quadratic cost functional L_ω $(\omega \in \Omega)$.

We now sketch the standard calculus of variations approach to finding solution u of the linear regulator problem. The idea is to convert the optimization problem in to a problem in Hamiltonian dynamics by considering the Hamiltonian "$H_\omega = \tilde{L}_\omega + \Sigma_i\, y_i \dot{x}_i$." More precisely, set

$$H_\omega(x,y,u) = \tilde{L}_\omega(x,u) + \langle y, A(\omega)x + B(\omega)u\rangle \tag{3.2}$$

for $x, y \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$. The reader may worry that "$A(\omega)$" is not strictly speaking well-defined; it will turn out that this does not matter as far as the final formulation of the problem is concerned. Note that

$$\begin{aligned} D_u H_\omega(x,y,u) &= Ru + B^*y \\ D_u^2 H_\omega(x,y,u) &= R, \end{aligned}$$

where D_u denotes the Fréchet derivative with respect to u. Since R is positive definite and symmetric, it is invertible, hence our Hamiltonian is *regular* and also it is convex.

At this point one applies the Pontryagin maximal principle [26] to conclude that a necessary condition on any control function minimizing L_ω is

$$D_u H_\omega = 0. \tag{3.3}$$

Condition (3.3) yields the *feedback rule*

$$u = -R^{-1}B^*y \tag{3.4}$$

Substituting $u = -R^{-1}B^*y$ in the formula for H_ω, one obtains

$$\begin{aligned} H_\omega(x,y) = {} & \tfrac{1}{2}\langle x, Q(\omega)x\rangle + \tfrac{1}{2}\langle y, A(\omega)x\rangle \\ & - \tfrac{1}{2}\langle y, B(\omega)R^{-1}(\omega)B^*(\omega)y\rangle \end{aligned}$$

for $\binom{x}{y} \in \mathbf{R}^{2n}$. One can now use general arguments involving the regularity and convexity of H_ω to show that the control u which optimizes L_ω is given by the feedback rule (3.4), where $y(t)$ is obtained from a certain trajectory $\binom{x(t)}{y(t)}$ of the Hamiltonian differential equation generated by H_ω and where $x(0) = x_0$.

This completes our sketch of the calculus of variations approach to finding an optimal solution u. We consider then the Hamiltonian system corresponding to H_ω defined above,

$$x' = \frac{\partial H_\omega}{\partial y}, \quad y' = -\frac{\partial H_\omega}{\partial x} \qquad (\omega \in \Omega).$$

These equations take the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(\tau_t(\omega)) & -BR^{-1}B^*(\tau_t(\omega)) \\ -Q(\tau_t(\omega)) & -A^*(\tau_t(\omega)) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad (3.5)_\omega$$

for each $\omega \in \Omega$. It will sometimes be convenient to write $z = \binom{x}{y}$ and

$$f(\omega) = \begin{pmatrix} A(\omega) & -BR^{-1}B^*(\omega) \\ -Q(\omega) & -A^*(\omega) \end{pmatrix}$$

for the coefficient matrix in $(3.5)_\omega$. Then Eq. $(3.5)_\omega$ becomes

$$z' = f(\tau_t(\omega))z \qquad (z \in \mathbf{R}^{2n}, \quad \omega \in \Omega). \qquad (3.6)_\omega$$

Observe that f takes values in the Lie algebra $\mathrm{sp}(n, \mathbf{R})$ of infinitesimally symplectic $2n \times 2n$ real matrices.

The function $u(t)$ which minimizes L_ω must lie in $L^2([0,\infty), \mathbf{R}^m)$. The feedback rule (3.4) suggests that we look for solutions $\binom{x(t)}{y(t)}$ of $(3.5)_\omega$ for which $x(0) = x_0$ and $y(t) \in L^2([0,\infty), \mathbf{R}^n)$. We will show that each Eq. $(3.5)_\omega$ admits a unique solution with these properties under mild controllability assumptions on Eqs. $(1.1)_\omega$. In fact we will actually show much more. We will show that Eq. $(3.6)_\omega$ admits an ED. Via rule (3.4), we will then obtain the solution of the random linear regulator problem. The theory of ED will then give us our robustness, stability and recurrence results.

Remark 3.1. Classically, one solves $(3.5)_\omega$ with $x(0) = x_0$ and $y(t) \in L^2([0,\infty), \mathbf{R}^n)$ by solving the Riccati equation corresponding to $(3.5)_\omega$. We use the theory of exponential dichotomy and do not solve the Riccati equation directly.

We now review the *Floquet theory* for Eqs. $(3.5)_\omega$ or $(3.6)_\omega$ as given in [15, 17]. A key result in this theory is a criterion (involving the rotation number) for equations $(3.6)_\omega$ to have an exponential dichotomy.

Let us define this latter concept. Let $\Phi(\omega, t)$ be the fundamental matrix solution of $(3.6)_\omega$ satisfying $\Phi(\omega, 0) = I$—the $2n \times 2n$ identity matrix. Then Φ: $\Omega \times \mathbf{R} \to \mathrm{Sp}(n, \mathbf{R})$ (the real symplectic group) is continuous and satisfies the cocycle identity:

$$\Phi(\omega, t+s) = \Phi(\tau_t(\omega), s)\Phi(\omega, t) \qquad (\omega \in \Omega, t, s \in \mathbf{R}).$$

DEFINITION 3.2. Equations $(3.6)_\omega$ are said to have an *exponential dichotomy* (ED for short) *over* Ω if there exist continuous vector subbundles V^s, V^u of $\mathbf{R}^{2n} \times \Omega$ with the following properties:

(a) $V^s \oplus V^u = \mathbf{R}^{2n} \times \Omega$ (Whitney sum).

(b) V^s and V^u are invariant under the "skew product flow" $\hat{\tau}$ on $\mathbf{R}^{2n} \times \Omega$ defined by

$$\hat{\tau}_t(v, \omega) = (\Phi(\omega, t)v, \tau_t(\omega)), \qquad ((v, \omega) \in \mathbf{R}^{2n} \times \Omega).$$

(c) There exist constants $K > 0$, $\alpha > 0$ such that

$$\text{if } (v, \omega) \in V^s \text{ then } \|\Phi(\omega, t)v\| \le Ke^{-\alpha t}\|v\| \qquad (t \ge 0), \qquad \text{and}$$
$$\text{if } (v, \omega) \in V^u \text{ then } \|\Phi(\omega, t)v\| \le Ke^{\alpha t}\|v\| \qquad (t \ge 0).$$

One of the main steps in the development of the Floquet theory is the introduction of a parameter λ (spectral parameter) in the following way. Let γ be a continuous, $2n \times 2n$ matrix-valued function on Ω such that

$$\gamma(\omega) = \gamma^*(\omega) \ge 0 \qquad (\omega \in \Omega).$$

Consider the family of differential equations

$$z' = [f(\tau_t(\omega)) + \lambda \mathbf{J}\gamma(\tau_t(\omega))]z, \tag{3.7$_{\omega,\lambda}$}$$

where $\mathbf{J} = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ is the usual skew-symmetric matrix of dimension $2n \times 2n$. These equations take the standard form (Atkinson [4])

$$\mathbf{J}^{-1}z' = [\mathbf{J}^{-1}f(\tau_t(\omega)) + \lambda\gamma(\tau_t(\omega))]z$$

upon multiplication by $\mathbf{J}^{-1}$; note that $\mathbf{J}^{-1}f(\cdot)$ is symmetric because $f(\cdot)$ takes values in $\mathrm{sp}(n, \mathbf{R})$. When discussing quantities related to Eqs. $(3.7)_{\omega,\lambda}$, we will omit reference to the parameter λ unless confusion would result.

The Floquet theory requires the following non-degeneracy assumption on equations $(3.6)_\omega$. It is closely related to the condition placed on Eq. $(3.7)_{\omega,\lambda}$ by Atkinson in [4, Chap. 9].

HYPOTHESIS (3.3). For each minimal subset $M \subseteq \Omega$, there exists at least one point $\omega_0 \in M$ with the following property. If v is a non-zero vector in $\mathbf{R}^{2n}$, then

$$\int_{-\infty}^{\infty} \left\| \gamma(\tau_t(\omega_0))\Phi(\omega_0, t)v \right\|^2 dt > 0.$$

Recall that $\Phi(\omega_0, t)$ is the fundamental matrix solution of $(3.6)_{\omega_0}$.

Alternatively, $\Phi(\omega_0, t)$ is obtained by solving $(3.7)_{\omega,\lambda}$ with $\lambda = 0$. It can be shown [17, Lemma (2.7)] that Hypothesis (3.3) actually implies that

$$\int_{-\infty}^{\infty} \left\| \gamma(\tau_t(\omega))\Phi_\lambda(\omega, t)v \right\|^2 dt > 0$$

for all $\omega \in M$ and all $\lambda \in \mathbf{C}$, whenever M is a minimal subset of Ω. Here $\Phi_\lambda(\omega, t)$ is of course, the fundamental matrix solution of Eq. $(3.7)_{\omega,\lambda}$.

We now discuss the rotation number $\alpha = \alpha(\lambda)$ for Eqs. $(3.7)_{\omega,\lambda}$. We need to review the rudiments of the structure of the set of *Lagrange planes* in $\mathbf{R}^{2n}$. This discussion will also help to clarify the geometric meaning of the solution of the Riccati equation associated with $(3.5)_\omega$.

So, let $\eta \subseteq \mathbf{R}^{2n}$ be a linear subspace of dimension n. Cal η a *Lagrange plane* if $\langle x, \mathbf{J}y\rangle = 0$ for all $x, y \in \eta$. Let $\mathscr{L}_\mathbf{R}$ be the compact manifold of all real Lagrange planes $\eta \subseteq \mathbf{R}^{2n}$.

Consider the open subset $U \subseteq \mathscr{L}_\mathbf{R}$ consisting of those Lagrange planes η which admit a basis of vectors of the form

$$\begin{pmatrix} \hat{e}_1 \\ m_1 \end{pmatrix}, \cdots \begin{pmatrix} \hat{e}_n \\ m_n \end{pmatrix}.$$

Here $\{\hat{e}_1, \cdots \hat{e}_n\}$ is the standard basis in $\mathbf{R}^n$ (not $\mathbf{R}^{2n}$), and $\{m_1, \cdots m_n\}$ are n-dimensional column vectors. Each $\eta \in U$ can be parametrized by the $n \times n$ real matrix $m = (m_1, \cdots m_n)$ whose columns are $m_1, \cdots m_n$. The fact that η is a Lagrange plane implies that m is symmetric: $m^* = m$.

Next let $\eta_0 = \mathrm{Span}\{e_{n+1}, \ldots, e_{2n}\}$ be the n-plane in $\mathbf{R}^{2n}$ spanned by the last n unit vectors. It is easily seen that $\eta_0 \in \mathscr{L}_\mathbf{R}$. Let

$$\mathscr{C} = \{\eta \in \mathscr{L}_\mathbf{R} \mid \dim(\eta \cap \eta_0) \geq 1\} \subseteq \mathscr{L}_\mathbf{R}.$$

The set $\mathscr{C}$ is called *Maslov cycle*. One can show that $\mathscr{L}_\mathbf{R} - \mathscr{C}$ is simply connected. The complement of $\mathscr{C}$ in $\mathscr{L}_\mathbf{R}$ is just the open set U defined above. Also $\mathscr{C}$ is of codimension 1 in $\mathscr{L}_{\mathbf{R}}$ in an appropriate sense (though it is not a submanifold of $\mathscr{L}_{\mathbf{R}}$, but rather a stratified submanifold). Furthermore, $\mathscr{C}$ is *two sided* in $\mathscr{L}_{\mathbf{R}}$ in the following sense: if an oriented curve in $\mathscr{L}_{\mathbf{R}}$ passes through $\mathscr{C}$, one can assign an oriented intersection number of each point of (transversal) intersection. This intersection number takes values between $-n$ and n (see [1]).

Fix $\lambda \in \mathbf{R}$ and again let $\Phi_\lambda(\omega, \tau)$ be the fundamental matrix solution of $(3.7)_{\omega,\lambda}$. Since $\lambda \in \mathbf{R}$, $\Phi_\lambda(\omega, t)$ lies in the symplectic group $\mathrm{Sp}(n, \mathbf{R})$ for all (ω, t). It follows that, if $\eta \in \mathscr{L}_{\mathbf{R}}$, then $\Phi_\lambda(\omega, t)\eta \in \mathscr{L}_{\mathbf{R}}$ where $\Phi_\lambda(\omega, t)\eta$ denotes the image of $\eta \subseteq \mathbf{R}^{2n}$ under the indicated linear transformation. Thus if $T > 0$, the map $\kappa: t \to \Phi_\lambda(\omega, t)\eta$ from $[0, T]$ to $\mathscr{L}_{\mathbf{R}}$ defines a curve in $\mathscr{L}_{\mathbf{R}}$.

Let $n(T)$ be the number of oriented intersections this curve makes with the Maslov cycle $\mathscr{C}$. We slide over certain details involved in giving a precise definition of $n(T)$; these matters are discussed in [15]. consider the limit

$$\alpha(\lambda) = \operatorname*{Lim}_{T \to \infty} \frac{\pi n(T)}{T}. \tag{3.8}$$

It exists in the following sense. Let μ be an *ergodic measure* on Ω (see the definition below). Then there is a subset $\Omega_1 \subseteq \Omega$, whose complement has μ-measure zero, such that the limit in (3.8) exists and is independent of both, the choice of $\omega \in \Omega_1$ and $\eta \in \mathscr{L}_{\mathbf{R}}$. We call this number $\alpha(\lambda) = \alpha(\lambda, \mu)$ (or rather the function $\lambda \to \alpha(\lambda)$) the *rotation number* of Eqs. $(3.7)_{\omega,\lambda}$ with respect to the ergodic measure μ. We see that $\alpha(\lambda)$ measures the average number of points of intersection of the curve $\kappa(t)$ with the Maslov cycle $\mathscr{C}$ as $T \to -\infty$. It is proved in [15] that $\lambda \to \alpha(\lambda)$ is continuous and monotone non-decreasing.

We pause to recall the definition of ergodic measure (see [24] for a detailed discussion).

DEFINITIONS 3.4. Let μ be a Radon probability measure on Ω. Then μ is *invariant* if, for each Borel subset $B \subseteq \Omega$, one has $\mu(\tau_t(B)) = \mu(B)$ for all $t \in \mathbf{R}$. The measure μ is *ergodic* if, measure of any invariant set is either 0 or 1. We also recall that the *topological support* of a Radon measure μ on Ω is the complement of the largest open set V satisfying $\mu(V) = 0$.

We now state the basic relation between the rotation number and the existence of ED for Eqs. $(3.7)_{\omega,\Lambda}$.

THEOREM 3.5. *Suppose that μ is an ergodic measure on Ω whose topological support is all of Ω. Further suppose that the Atkinson-type Hypothesis* (3.3) *is valid. Let $I \subseteq \mathbf{R}$ be an open interval. Then Eqs.* $(3.7)_{\omega,\lambda}$

have an exponential dichotomy over Ω *for all* $\lambda \in I$ *if and only if the rotation number* $\lambda \to \alpha(\lambda)$ *is constant on* I.

The theorem is proved in [17]. Since α is continuous and monotone, the assumption of constancy of α on I is equivalent to equality of the values of α at the end points of I.

We finish Section 3 by returning for a moment to the Atkinson-type condition (3.3). We show that it is actually equivalent to a controllability condition.

PROPOSITION 3.6. *Hypothesis* (3.3) *holds if and only if the control process*

$$z' = -f^*(\tau_t(\omega))z + \gamma(\tau_t(\omega))u \qquad (3.9)_\omega$$

is (*controllable and hence*) *uniformly controllable on each minimal subset* $M \subseteq \Omega$.

Proof. Consider first the "only if" implication. Write out the condition in (3.3) and note that $\Phi^*(\omega_0, t)^{-1}$ is the fundamental matrix solution of the adjoint system

$$z' = -f^*(\tau_t(\omega))z.$$

Using symmetry of γ and Lemma (2.4), we see that $(3.9)_\omega$ is uniformly controllable on each minimal subset $M \subseteq \Omega$.

The "if" implication follows quickly from the non-singularity criterion of the controllability matrix. ▮

We see that there is an interesting interplay between the notion of uniform controllability, the Atkinson condition, and exponential dichotomy. Palmer [25] has considered the relation between controllability, bounded input-bounded output stability and the existence of exponential dichotomy.

4. SOLUTION OF THE RANDOM FEEDBACK STABILIZATION PROBLEM

We begin by introducing the basic controllability assumptions which will imply that the random regulator problem and the random feedback control problem are solvable. These assumptions are natural variants of the classical conditions found in [7, 14, 20]. Define

$$C(\omega) = \sqrt{Q(\omega)} \qquad (\omega \in \Omega)$$

to be the unique positive semidefinite square root of the matrix function $Q(\omega)$.

HYPOTHESES 4.1. Suppose that, for each minimal subset $M \subseteq \Omega$, there exists at least one $\omega_0 \in M$ such that:

(a) $x' = -A^*(\tau_t(\omega_0))x + C(\tau_t(\omega_0))u_1$ is controllable;

(b) $y' = A(\tau_t(\omega_0))y + B(\tau_t(\omega_0))u_2$ is controllable.

For later convenience we write u_1, u_2 in place of u in Hypotheses 4.1. Note that Hypothesis 4.1(a) is automatically satisfied if $Q(\omega)$ is positive definite for all $\omega \in \Omega$.

We first show that Hypotheses 4.1 implies the Atkinson-Hypothesis 3.3 for appropriate γ. Define

$$\gamma(\omega) = \begin{pmatrix} Q(\omega) & 0 \\ 0 & B(\omega)R^{-1}(\omega)B^*(\omega) \end{pmatrix}$$

so that γ is continuous, symmetric and positive semi-definite. Consider the family of control processes $(3.9)_\omega$ encountered at end of Section 3; explicitly,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -A^*(\tau_t(\omega)) & Q(\tau_t(\omega)) \\ (BR^{-1}B^*)(\tau_t(\omega)) & A(\tau_t(\omega)) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$+ \lambda\begin{pmatrix} Q(\tau_t(\omega)) & 0 \\ 0 & (BR^{-1}B^*)(\tau_t(\omega)) \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \qquad (4.1)_\omega$$

LEMMA 4.2. *Hypotheses* (4.1) *implies that, if $M \subseteq \Omega$ is minimal, then the control systems* $(4.1)_\omega$ *are uniformly controllable for each* $\omega \in M$.

COROLLARY 4.3. *If Hypotheses* 4.1 *holds, then Hypothesis* 3.3 *is valid for Eqs*. $(3.7)_{\omega,\lambda}$ *with*

$$f(\omega) = \begin{pmatrix} A(\omega) & -BR^{-1}B^*(\omega) \\ -Q(\omega) & -A^*(\omega) \end{pmatrix}$$

and

$$\gamma(\omega) = \begin{pmatrix} Q(\omega) & 0 \\ 0 & BR^{-1}B^*(\omega) \end{pmatrix}.$$

Corollary 4.3 follows from Proposition 3.6 and Lemma 4.2. It will allow us to apply Theorem 3.5 to Eqs. $(3.7)_{\omega,\lambda}$.

Proof of Lemma 4.2. Let $\Psi(\omega, t)$ be the fundamental matrix solution of $x' = -A^*(\tau_t(\omega))x$, $(\omega \in \Omega)$. The nonsingularity (and hence the positive definiteness) of the controllability matrix yields positive numbers $T > 0$, $\delta > 0$ such that

$$\int_0^T \left\| C(\tau_t(\omega_0))\Psi(\omega_0, t)v \right\|^2 dt \geq \delta \|v\|^2$$

for all $v \in \mathbf{R}^n$. Using symmetry and semi-definiteness of C together with a compactness argument, one can find $\tilde{\delta} > 0$ such that

$$\int_0^T \left\| Q(\tau_t(\omega_0))\Psi(\omega_0, t)v \right\|^2 dt \geq \tilde{\delta} \|v\|^2$$

for all $v \in \mathbf{R}^n$. By Lemma 2.4, we see that the control systems

$$x' = -A^*(\tau_t(\omega))x + Q(\tau_t(\omega))u_1 \qquad (4.2)_\omega$$

are uniformly controllable for all $\omega \in M$.

Using strict positive definiteness of R and an argument like that just given, one shows that, if (4.1)(b) holds, then the control systems

$$y' = A(\tau_t(\omega))y + B(\tau_t(\omega))R^{-1}(\tau_t(\omega))B^*(\tau_t(\omega))u_2 \qquad (4.3)_\omega$$

are uniformly controllable over M.

Let us return again to the point $\omega_0 \in M$. Let $\binom{x_0}{y_0} \in \mathbf{R}^{2n}$. There exists a number $T > 0$ and controls $u_1, u_2{:}[0, T] \to \mathbf{R}^m$ with the following properties:

(i) the solution $x(t)$ of $(4.2)_{\omega_0}$ with $x(0) = x_0$ satisfies $x(T) = 0$;

(ii) the solution $y(t)$ of $(4.3)_{\omega_0}$ with $y(0) = y_0$ satisfies $y(T) = 0$.

Define

$$w_1(t) = u_1(t) - x(t)$$
$$w_2(t) = u_2(t) - y(t)$$

for $0 \leq t \leq T$. The control $\binom{w_1}{w_2}$ steers $\binom{x_0}{y_0}$ to zero in time T for Eq. $(4.1)_{\omega_0}$. This shows that the process $(4.1)_{\omega_0}$ is controllable. By Lemma 2.4, we obtain the statement of Lemma 4.2. ∎

We are now in a position to apply Theorem 3.5. Indeed, we now consider the rotation number of Eqs. $(3.7)_{\omega,\lambda}$ with f, γ as above.

LEMMA 4.4. *Assume that Hypotheses* 4.1 *holds. Let* $I = (-1/2, 1/2)$. *If* $\lambda \in I$ *and* $M \subseteq \Omega$ *is a minimal set, then the rotation number of Eqs.* $(3.7)_{\omega,\lambda}$

equals zero for every ergodic measure μ *whose topological support is contained in* M.

Proof. Let $\omega \in M$. Consider the boundary value problem

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \left[\begin{pmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{pmatrix}(\tau_t(\omega)) + \lambda \mathbf{J}\begin{pmatrix} Q & 0 \\ 0 & (BR^{-1}B^*) \end{pmatrix}(\tau_t(\omega))\right]\begin{pmatrix} x \\ y \end{pmatrix} \qquad (4.4)_\omega$$

$$x(0) = x(T) = 0$$

where T is some positive number. We show that this boundary value problem has only the trivial solution if T is sufficiently large.

To do so, let $\binom{x(t)}{y(t)}$ be a solution of $(4.4)_\omega$. Then

$$0 = \langle x(T), y(T)\rangle - \langle x(0), y(0)\rangle = \int_0^T \frac{d}{dt}\langle x(t), y(t)\rangle \, dt$$

$$= \int_0^T \left[\left\langle \frac{d}{dt}x(t), y(t)\right\rangle + \left\langle x(t), \frac{d}{dt}y(t)\right\rangle\right] dt$$

$$= \int_0^T \left[\langle Ax - (1+\lambda)BR^{-1}B^*y, y\rangle + \langle x, (\lambda - 1)Qx - A^*y\rangle\right] dt.$$

Hence, $0 = (\lambda - 1)\int_0^T \|Cx\|^2\, dt - (1+\lambda)\int_0^T \|R^{-(1/2)}B^*y\|^2\, dt$, that is,

$$(\lambda - 1)\int_0^T \|Cx\|^2\, dt = (1+\lambda)\int_0^T \|R^{-(1/2)}B^*y\|^2\, dt.$$

Since $\lambda \in (-1/2, 1/2)$, we must have $C(\tau_t(\omega))x(t) = 0 = B^*(\tau_t(\omega))y(t)$ for all $0 \le t \le T$.

We conclude that $y'(t) = -A^*(\tau_t(\omega))y(t)$. Hence using uniform controllability of $y' = Ay + Bu$ (see Lemma (2.4) and the relation

$$\int_0^T \|B^*y\|^2\, dt = 0,$$

we see that $y(t) = 0$ on $[0, T]$ if T is sufficiently large. Since $x(0) = 0$, we have by uniqueness that $x(t)$ is identically zero on $[0, T]$ as well.

Now we can show that $\alpha(\lambda, \mu) = 0$ if $\lambda \in I$. Consider the Lagrange plane $\eta_0 = \{\binom{0}{y} \mid y \in \mathbf{R}^n\}$ used in defining the Maslov cycle $\mathscr{C}$. Note that, if $\dim(\Phi_\lambda(\omega, T)\eta_0 \cap \eta_0) \ge 1$ for some $T > 0$ and some $\lambda \in I$, then there exists $0 \neq y_0 \in \mathbf{R}^n$ such that

$$\Phi_\lambda(\omega, T)\begin{pmatrix} 0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ y(T) \end{pmatrix}.$$

But then $y(T) = 0$ by the preceding paragraph, a contradiction if T is large. Hence for all $\lambda \in I$ and all large T, the Lagrange plane $\Phi_\lambda(\omega, T)\eta_0$ does not lie on the Maslov cycle. By (3.8), we must have $\alpha(\lambda) = \alpha(\lambda, \mu) = 0$ for all $\lambda \in I$. ∎

LEMMA 4.5. *Equations* $(3.5)_\omega$ *admit an exponential dichotomy over* Ω. *In other words, Eqs.* $(3.7)_{\omega, \lambda}$ *admit an exponential dichotomy over* Ω *when* $\lambda = 0$.

Proof. If M is a minimal subset of Ω, then Eqs. $(3.5)_\omega$ admit an ED over M by Lemma 4.4.

By Theorem (3.1) of [15], the dimensions of the stable and unstable bundles V^s, V^u over M are each equal to n. That is, these dimensions are independent of the minimal subset $M \subseteq \Omega$.

Next let $\omega \in \Omega$ and suppose that $\binom{x(t)}{y(t)}$ is a solution of Eq. $(3.5)_\omega$ which is bounded on all of $\mathbf{R}$. We claim that $x(t)$ and $y(t)$ are identically zero. To prove this, note first that there are sequences $t_n \to \infty$, $s_n \to -\infty$ such that $\binom{x(t_n)}{y(t_n)} \to \binom{0}{0}$ and $\binom{x(s_n)}{y(s_n)} \to \binom{0}{0}$. For if, for example, there was no such sequence $\{s_n\}$, then it is easily seen that, for each $\overline{\omega}$ in the α-limit set of ω, the equation $(3.5)_{\overline{\omega}}$ would admit a non-trivial solution bounded on all of $\mathbf{R}$. But the α-limit set of ω contains a minimal set M. Since Eqs. $(3.5)_\omega$ have ED on M, there can be no non-trivial bounded solution of $(3.5)_{\overline{\omega}}$ if $\overline{\omega} \in M$. This contradiction proves that $\{s_n\}$ exists, and in a similar way one proves that $\{t_n\}$ exists. Now,

$$\langle x(t_n), y(t_n)\rangle - \langle x(s_n), y(s_n)\rangle = \int_{s_n}^{t_n} \frac{d}{dt}\langle x(t), y(t)\rangle\, dt$$

$$= -\int_{s_n}^{t_n}\left[\|R^{-(1/2)}B^*y\|^2 + \|Cx\|^2\right] dt.$$

We conclude that $B^*(\tau_t(\omega))y(t) = 0 = C(\tau_t(\omega))x(t)$ for all $t \in \mathbf{R}$. But now using the controllability condition (Hypothesis 4.1a, b) together with Proposition 2.5 and arguing as in the proof of Lemma (4.4), we see that $x(t)$ and $y(t)$ are identically zero.

We now apply a theorem of [27], the proof of which is based on ideas of Conley [8]. This theorem states that if the dimensions of V^s, V^u are independent of M and if no equation $(3.5)_\omega$ admits a non-trivial bounded solution, then Eqs. $(3.5)_\omega$ admit an ED over Ω. This completes the proof of Lemma 4.5. ∎

We need one more bit of information before turning to the solution of the linear regulator problem and the feedback control problem. Let $V^s \subseteq \mathbf{R}^{2n} \times \Omega$ be the stable bundle of Eqs. $(3.5)_\omega$.

LEMMA 4.6. *Let $\omega \in \Omega$ and let $\binom{x}{y} \in V^s(\omega) = (\mathbf{R}^{2n} \times \{\omega\}) \cap V^s$. Then $\langle x, y \rangle \neq 0$ unless $x = y = 0$.*

Proof. We essentially repeat part of the proof of Lemma 4.5. Suppose that there exists $\binom{x_0}{y_0} \in V^s(\omega)$ with $\langle x_0, y_0 \rangle = 0$. Let $\binom{x(t)}{y(t)}$ be the solution of $(3.5)_\omega$ with initial value $\binom{x_0}{y_0}$. Then

$$\begin{aligned}\langle x(T), y(T) \rangle &= \int_0^T \frac{d}{dt} \langle x(t), y(t) \rangle \, dt \\ &= -\int_0^T \|R^{-(1/2)} B^* y\|^2 \, dt - \int_0^T \|Cx\|^2 \, dt.\end{aligned}$$

Since $\binom{x_0}{y_0} \in V^s(\omega)$ we have $x(T) \to 0$, $y(T) \to 0$ as $T \to \infty$. Arguing as in the proof of Lemma 4.5, we now show that $x(t) \equiv 0 \equiv y(t)$, thus $x_0 = y_0 = 0$. This completes the proof. ∎

By a result of [15], $V^s(\omega)$ is a Lagrange plane for each $\omega \in \Omega$ and Lemma 4.6 implies that, for each $\omega \in \Omega$, the projection of $V^s(\omega)$ on to $\eta_1 = \{\binom{x}{0} \mid x \in \mathbf{R}^n\}$ is on to. Thus $V^s(\omega)$ has a basis of the form

$$\begin{pmatrix} \hat{e}_1 \\ m_1(\omega) \end{pmatrix}, \cdots \begin{pmatrix} \hat{e}_n \\ m_n(\omega) \end{pmatrix};$$

i.e., can be parametrized by the following (symmetric) matrix:

$$m(\omega) = (m_1(\omega), \ldots, m_n(\omega))$$

(see Section 3). We see that, if $\binom{x_0}{y_0} \in V^s(\omega)$ and $\binom{x(t)}{y(t)}$ is the corresponding solution of $(3.5)_\omega$, then $y(t) = m(\tau_t(\omega))x(t)$.

The continuity of m in ω follows from the continuous variation of the fibers $V^s(\omega)$. By invariance of V^s under the flow $\hat{\tau}$ on $\mathbf{R}^{2n} \times \Omega$ (see Definition 3.2), we see that the function $t \to m_t(\omega)$ satisfies the Riccati equation

$$m' = -A^*m - mA + mBR^{-1}B^*m - Q.$$

We are ready to solve the linear regulator problem. Given any control $u(t)$, let $x(t)$ be the corresponding solution of

$$\begin{aligned} x' &= A(\tau_t(\omega))x + B(\tau_t(\omega))u \\ x(0) &= x_0. \end{aligned}$$

The following formula can be verified using the Riccati equation for m:

$$\begin{aligned}\frac{d}{dt}&\langle m(\tau_t(\omega))x(t), x(t)\rangle \\ &= \left\| R^{1/2}(\tau_t(\omega))\left[u(t) + R^{-1}B^* m(\tau_t(\omega))x(t)\right]\right\|^2 \\ &\quad - 2\tilde{L}_\omega(x(t), u(t)).\end{aligned} \tag{4.5}$$

Thus L_ω is minimized by choosing

$$u(t) = -R^{-1}(\tau_t(\omega))B^*(\tau_t(\omega))m(\tau_t(\omega))x(t).$$

The corresponding $x(t)$ decreases exponentially as $t \to \infty$ and hence $u(t)$ is square-integrable on $[0, \infty)$. The minimum value of the functional L_ω is

$$\begin{aligned}\int_0^\infty 2\tilde{L}_\omega(x(t), u(t))\, dt &= -\int_0^\infty \frac{d}{dt}\langle m(\tau_t(\omega))x(t), x(t)\rangle\, dt \\ &= \langle m(\omega)x_0, x_0\rangle.\end{aligned}$$

This shows that $u(t)$ is the unique control solving the random regulator problem.

Now we turn to the random feedback stabilization problem. Define

$$K(\omega) = -R^{-1}(\omega)B^*(\omega)m(\omega) \qquad (\omega \in \Omega).$$

Clearly K is continuous in ω. Let $\binom{x(t)}{m(\tau_t(\omega))x(t)}$ be the unique solution of $(3.5)_\omega$ lying in the stable bundle V^s which satisfies $x(0) = x_0$. We see from $(3.5)_\omega$ that $x(t)$ satisfies the closed-loop system

$$x' = \left[A(\tau_t(\omega)) + B(\tau_t(\omega))K(\tau_t(\omega))\right]x. \tag{4.6}$$

Since $x(t) \to 0$ exponentially as $t \to \infty$, the function K does indeed stabilize $(1.1)_\omega$ for each $\omega \in \Omega$. This solves the feedback stabilization problem.

At this point we note that robustness of K and smoothness of K with respect to parameters follow from the corresponding properties for the bundle V^s and V^u. For a very general smoothness result, see Yi [28]. As a sample of the type of robustness result that can be obtained, we explain briefly the implications of a result of Coppel [9, p. 34].

Let A_0 and B_0 be constant matrices for which the controllability assumptions (4.1) are satisfied. For convenience we choose $C = \mathrm{Id}$ (the identity) in this discussion. Let K_0 be the corresponding feedback matrix. We view A_0 (respectively B_0) as an element of $L^p_{\mathrm{loc}}(\mathbf{R}, M(n,n))$ (respectively $L^p_{\mathrm{loc}}(\mathbf{R}, M(n,m)))$ where $p > 1$. There exists numbers $\delta_0 > 0$, $L > 0$ with the following property. Let $A_1 \in L^p_{\mathrm{loc}}(\mathbf{R}, M(n, n))$, $B_1 \in L^p_{\mathrm{loc}}(\mathbf{R}, M(n,m))$ be functions such that

$$\operatorname*{Sup}_{t\in\mathbf{R}} \int_t^{t+1} \| A_1(s) \|^p \, ds \le \delta \le \delta_0 \quad \text{and} \quad \operatorname*{Sup}_{t\in\mathbf{R}} \int_t^{t+1} \| B_1(s) \|^p \, ds \le \delta \le \delta_0 .$$

Put $a(t) = A_0 + A_1(t)$, $b(t) = B_0 + B_1(t)$. Then there exists a feedback matrix $k(t)$ such that $x' = [a(t) + b(t)k(t)]x$ is exponentially stable and such that

$$\| k(t) - K_0 \| < L\delta \qquad (t \in \mathbf{R}).$$

Note that, if $Q(\omega)$ is strictly positive definite for all ω, then Hypothesis (4.1)(a) is automatically satisfied. Thus the possibility of solving the feedback stabilization problem really depends only on the verification of Hypothesis (4.1)(b), i.e., on controllability of $(1.1)_\omega$.

Finally, we consider the question of pole relocation. For autonomous systems $x' = ax + bu$, this refers to the possibility of choosing Q, R in such a way that the resulting feedback K has the property that the eigenvalues of $a + bK$ take on prescribed values in the left half-plane. We wish to consider the random version of pole relocation.

Recall that a real number β is a *Lyapunov exponent* of (4.6) if there is a non-zero vector $x_0 \in \mathbf{R}^n$ such that, if $x(t)$ is the solution of (4.6) with $x(0) = x_0$, then

$$\beta = \overline{\lim_{t\to\infty}} \frac{1}{t} \log\| x(t) \|.$$

We are going to show that $K(\omega)$ can be chosen so that all Lyapunov exponents of (4.6) lie as far to the left of the origin as desired.

To do so, choose $\delta > 0$ and consider the random regulator problem with cost functional

$$L_\omega(x,u) = \int_0^\infty \left(\delta\|x\|^2 + \|u\|^2 \right) dt.$$

Clearly Hypotheses 4.1 are valid. The m-function is positive definite because if $\langle m(\omega)x_0, x_0 \rangle = 0$ for some $\omega \in \Omega$ and $0 \ne x_0 \in \mathbf{R}^n$, then by (4.5)

$$-\langle m(\tau_T(\omega))x(T), x(T) \rangle = \int_0^T \left[\delta\|x\|^2 + \|B^* m x\|^2 \right] dt,$$

where $x(t)$ is the solution of (4.6) with $x(0) = x_0$. The left-hand side tends to zero as $T \to \infty$, hence $x(t) = 0$ for al t, hence $x_0 = 0$.

Now let ε_1 and ε_2 be respectively the minimum and the maximum of the set of numbers $\{\langle m(\omega), x, x\rangle \mid \omega \in \Omega, \|x\| = 1\}$. Then

$$0 < \varepsilon_1 \|x\|^2 \leq \langle m(\omega)x, x\rangle \leq \varepsilon_2 \|x\|^2 \qquad \text{forall } x \in \mathbf{R}^n.$$

Thus

$$\frac{d}{dt}\langle m(\tau_t(\omega))x(t), x(t)\rangle \leq -\delta\|x\|^2 \leq -\frac{\delta}{\varepsilon_2}\langle m(\tau_t(\omega))x(t), x(t)\rangle.$$

Therefore $\langle m(\tau_t(\omega))x(t), x(t)\rangle \leq Me^{-(\delta t/\varepsilon_2)}$ where M is some positive constant. Hence,

$$\|x(t)\|^2 \leq \frac{1}{\varepsilon_1}\langle m(\tau_t(\omega))x(t), x(t)\rangle \leq \frac{M}{\varepsilon_1}e^{-(\delta t/\varepsilon_2)}.$$

Thus every solution of the feedback system decays exponentially as $t \to \infty$ with rate at least $(-\delta/2\varepsilon_2)$, $(\omega \in \Omega)$. Thus the Lyapunov exponents lie always to the left of $(-\delta/2\varepsilon_2)$, uniformly in $\omega \in \Omega$. This last assertion is not always true if one has (unbounded) stochastic coefficients.

To summarize: we used the rotation number to show that Eq. $(3.5)_\omega$ has ED. This allowed us to solve the random feedback stabilization problem and obtain robustness properties, smoothness properties, preservation of recurrence of the solution, and the random pole relocation result.

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